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Mirror symmetry and generalized complex manifolds Part I. The transform on vector bundles, spinors, and branes

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Abstract

In this paper we begin the development of a relative version of T-duality in generalized complex geometry which we propose as a manifestation of mirror symmetry. Let M be an n-dimensional smooth real manifold, V a rank n real vector bundle on M, and ∇ a flat connection on V. We define the notion of a ∇-semi-flat generalized almost complex structure on the total space of V. We show that there is an explicit bijective correspondence between ∇-semi-flat generalized almost complex structures on the total space of V and $\nabla^{\vee}$-semi-flat generalized almost complex structures on the total space of $V^{\vee}$. We show that semi-flat generalized complex structures give rise to a pair of transverse Dirac structures on the base manifold. We also study the ways in which our results generalize some aspects of T-duality such as the Buscher rules. We show explicitly how spinors are transformed and discuss the induces correspondence on branes under certain conditions.

Keywords: Mirror symmetry; T-duality; 14J32; Generalized complex manifolds; Dirac manifolds

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1. Introduction

Mirror symmetry is often thought of as relating the very different worlds of complex geometry and symplectic geometry. It was recently shown by Hitchin [14] that symplectic and complex structures on a manifold have a simple common generalization called a generalized complex structure. This is a complexified version of Dirac geometry [9] along with an extra non-degeneracy condition. It is expected that mirror symmetry should give rise to an involution on sectors of the moduli space of all generalized complex manifolds of a fixed dimension. One of the most concrete descriptions of the mirror correspondence is the Strominger–Yau–Zaslow picture [24] in which mirror symmetry is interpreted as a relative T-duality along the fibers of a special Lagrangian torus fibration. This is sometimes referred to as "T-duality in half the directions". In our previous work [2], we investigated the linear algebraic aspects of T-duality for generalized complex structures. See also [27] for the analogous story in Dirac geometry. In this paper, we go one step further and construct an explicit mirror involution on certain moduli of generalized complex manifolds. Similarly to the case of Calabi–Yau manifolds the definition of our mirror involution depends on additional data. In our set up, we will consider generalized complex manifolds equipped with a compatible torus fibration. This involution, when applied to such a manifold, gives another with the same special properties, which we propose to identify as its mirror partner. In the special cases of a complex or symplectic structure on a semi-flat Calabi–Yau manifold our construction reproduces the standard T-duality of [20,22,23]. In addition we get new examples of mirror symmetric generalized complex manifolds, e.g. the ones coming from B-field transforms of complex or symplectic structures.

If V is a real vector space then [14] a generalized complex structure on V is a complex subspace $E \subseteq (V \oplus V^{\vee}) \otimes \mathbb{C}$ that satisfies $E \cap \overline{E} = (0)$ and is maximally isotropic with respect to the canonical quadratic form on $(V \oplus V^{\vee}) \otimes \mathbb{C}$. Let

$$f : V \oplus V^{\vee} \to W \oplus W^{\vee}$$

be a linear isomorphism which is compatible with the canonical quadratic forms. Then f induces a bijection between generalized complex structures on V and generalized complex structures on W. Transformations of this type can be viewed as linear analogues of the T-duality transformations investigated in the physics literature (see [17,26] and references therein). Mathematically they were studied in [27] for Dirac structures and in [2] for generalized complex structures. In this paper, the relevant case is where $V = A \oplus B$, $W = A^{\vee} \oplus B$, and $f : A \oplus B \oplus A^{\vee} \oplus B^{\vee} \to A^{\vee} \oplus B \oplus A \oplus B^{\vee}$ is the obvious shuffle map.

A generalized complex structure on a manifold X is a maximally isotropic sub-bundle of $(T_X \oplus T_X^{\vee}) \otimes \mathbb{C}$ that satisfies $E \cap \overline{E} = (0)$ and that E is closed under the Courant Bracket. In this paper, we shall preform a relative version of this T-duality for pairs of manifolds that are fibered over the same base and where the two fibers over each point are "dual" to each other. In other words we will find a way to apply the linear ideas above to the torus fibered approach. On each fiber, this process will agree with the linear map described above.

Throughout the paper as well as in Part II we comment on how our results relate to some of the well established results and conjectures of mirror symmetry [19,20,22–24] and also what they say in regards to the new developments in generalized Kähler geometry [12] and the relationships between generalized complex geometry and string theory [12,17,26] which have appeared recently. As mentioned in [17] we may interpret these dualities as being a generalization of the duality between the *A*-model and *B*-model in topological string theory. In the generalized Kähler case, they can be interpreted as dualities of supersymmetric nonlinear sigma models [11]. To this end, in Section 5 we sketch a relationship between branes in the sense of [12,17] in a semi-flat generalized complex structure and branes in its mirror structure. For some simple examples of branes, we give the relationship directly. We also show in Section 4 that the Buscher rules [5,6] for the transformation of metric and *B*-field hold between the mirror pairs of generalized Kähler manifolds that we consider.

It will be very interesting to extend the discussion in Section 5 to a full-fledged Fourier–Mukai transform on generalized complex manifolds. Unfortunately, the in-depth study of branes in generalized complex geometry is obstructed by the complexity of the behavior of sub-manifolds with regards to a generalized complex structure. Several subtle issues of this nature were analyzed in our previous paper [2]. In particular we investigated in detail the theory of sub and quotient generalized complex structures, described a zoo of sub-manifolds of generalized complex manifolds and studied the relations among those. We also gave a classification of linear generalized complex structures and constructed a category of linear generalized complex structures which is well adapted to the question of quantization. In a future work we plan to incorporate the structure of a torus bundle in this analysis and construct a complete Fourier transform for branes.

For the benefit of the reader who may not be familiar with generalized complex geometry, we have included Section 2 which introduces the linear algebra and some basics on generalized complex manifolds. More details on these basics may be found in [2,12,14–16].

2. Notation, conventions, and basic definitions

Overall, we will retain the notation and conventions from our previous paper [2], and so we only recall the most important facts for this paper as well as some changes. The dual of a vector space V will be denoted as $V^\vee$. We will often use the annihilator of a subspace $W \subseteq V$, which we will denote

$$\mathrm{Ann}(W) = \{f \in V^\vee | f|_W \equiv 0\} \subseteq V^\vee.$$

We will need the pairing $\langle \bullet, \bullet \rangle$ on $V \oplus V^\vee$, given by (following [15])

$$\langle v + f, w + g\rangle = -\tfrac{1}{2}(f(w) + g(v)) \text{ for all } v, w \in V,\ f, g \in V^\vee.$$

Given $v \in V$ and $f \in V^\vee$, we will write either $\langle f|v\rangle$ or $\langle v|f\rangle$ for $f(v)$. This pairing corresponds to the quadratic form $Q(v + f) = -f(v)$.

We will tacitly identify elements $B \in \bigwedge^2 V^\vee$ with linear maps $V \to V^\vee$. When thought of in this way, we have that the map is skew-symmetric: $B = -B^\vee$.

We will often consider linear maps of $V \oplus W \to V' \oplus W'$. Sometimes, these be written as matrices

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

with the understanding that $T_1 : V \to W'$, $T_2 : W \to V'$, $T_3 : V \to W'$ and $T_4 : W \to W'$ are linear maps. All of these conventions will be extended to vector bundles and their sections in the obvious way.

If M is a manifold, we let C_M^∞ denote the sheaf of *real-valued* C^∞ functions on M. We will use the same notation for a vector bundle and for its sheaf of sections. The tangent and cotangent bundles of M will be denoted by T_M and $T_M^\vee$. For a vector bundle V over a manifold M and a smooth map $f : N \to M$, we denote the pullback bundle by f^*V. A section of f^*V which is a pullback of a section e of V will be denoted $f^*(e)$. If f is an isomorphism onto its image or the projection map of a fiber bundle, the sections of this form give the sub-sheaf $f^{-1}V \subseteq f^*V$. We will sometimes replace $\bigwedge^\bullet T_M^\vee$ by $\Omega_M^\bullet$.

Now we will give some basic facts on generalized complex geometry that we will need in the paper. For more information the reader may see [2,12,14].

2.1. Generalized almost complex manifolds

Let M be a real manifold. A *generalized almost complex structure* on a real vector bundle $V \to M$ has been defined [12,14,15] in the following equivalent ways:

- A sub-bundle $E \subseteq V_{\mathrm{C}} \oplus V_{\mathrm{C}}^\vee$ which is maximally isotropic with respect to the standard pairing $\langle \bullet, \bullet \rangle$ and satisfies $E \cap \overline{E} = 0$;
- An automorphism $\mathcal{J}$ of $V \oplus V^\vee$ which is orthogonal with respect to $\langle \bullet, \bullet \rangle$ and satisfies $\mathcal{J}^2 = -1$.

Example 2.1. Let V be a real vector bundle.

(a) Let J be an almost complex structure on V. Then

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^\vee \end{pmatrix}$$

is a generalized almost complex structure on V. If $\mathcal{J}$ is a generalized complex structure on V that can be written in this form, we say that $\mathcal{J}$ is of *complex type*.

(b) Let ω be an almost symplectic form on V (i.e., a non-degenerate section ω of $\bigwedge^2 V^\vee$). Then

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

is a generalized complex structure on V. We say that such a $\mathcal{J}$ is of *symplectic type*.

There is also a way of describing generalized almost complex structures on V in terms of line sub-bundles of $\bigwedge V^{\vee} \otimes \mathbb{C}$ or spinors. This interpretation is very convenient for some purposes.

Definition 2.2 (*Gualtieri* [12] *and Hitchin* [15]). Let $\mathcal{J}$ be a generalized almost complex structure on a vector bundle V over M. Define the *canonical bundle* to be the complex line bundle $L \subseteq \bigwedge^{\bullet} V^{\vee} \otimes \mathbb{C}$ consist of the sections ϕ satisfying $\iota_v\phi + \alpha \wedge \phi$ for all sections $v + \alpha$ of the $+i$ eigenbundle E corresponding to the generalized almost complex structure on V. Sections of L will be called *representative spinors*.

For the case of an almost symplectic manifold with two-form ω, this line bundle is generated by $\exp(-\mathrm{i}\omega)$. For an almost complex manifold, one gets the usual canonical bundle. Spinor bundles can also be understood intrinsically in terms of the sheaves of modules over appropriate sheaf of Clifford algebras. The sections will satisfy certain restrictions over each fiber. They are known as pure spinors [8,12,15]. We have listed some of their features and examined their restriction to sub-manifolds in [2].

Definition 2.3 (*Hitchin [15]*). In the special case that $V = T_M$ has a generalized almost complex structure, we call M a *generalized almost complex manifold*.

In this case the spinor sections are differential forms. Such a manifold is always even dimensional as a real manifold. This can be shown by constructing two almost complex structures on M out of the generalized almost complex structure [15]. This also follows from the classification of generalized complex structures on a vector space which was done in our previous paper [2]. For the case of manifolds, a local structure theorem for generalized complex manifolds has been proven by Gualtieri [12].

Consider a real vector bundle V and an automorphism $\mathcal{J}$ of $V \oplus V^{\vee}$, written in matrix form as

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_1 & \mathcal{J}_2 \\ \mathcal{J}_3 & \mathcal{J}_4 \end{pmatrix}.$$

Let us record, for future use, the restrictions on the $\mathcal{J}_i$ coming from the conditions that $\mathcal{J}$ preserves the pairing $\langle \bullet, \bullet \rangle$ and satisfies $\mathcal{J}^2 = -1$. They are:

$$\mathcal{J}_1^2 + \mathcal{J}_2\mathcal{J}_3 = -1, \tag{2.1}$$

$$\mathcal{J}_1\mathcal{J}_2 + \mathcal{J}_2\mathcal{J}_4 = 0, \tag{2.2}$$

$$\mathcal{J}_3\mathcal{J}_1 + \mathcal{J}_4\mathcal{J}_3 = 0, \tag{2.3}$$

$$\mathcal{J}_4^2 + \mathcal{J}_3\mathcal{J}_2 = -1, \tag{2.4}$$

$$\mathcal{J}_4 = -\mathcal{J}_1^{\vee}, \tag{2.5}$$

$$\mathcal{J}_2^\vee = -\mathcal{J}_2, \tag{2.6}$$

$$\mathcal{J}_3^\vee = -\mathcal{J}_3. \tag{2.7}$$

2.2. *B- and β-field transforms*

Consider a real vector bundle V and a global section B of $\bigwedge^2 V^\vee$ [12–15]. Consider the transformation of $V \oplus V^\vee$

$$\exp(B) := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}.$$

It is easy to see that $\exp(B)$ is an orthogonal automorphism of $V \oplus V^\vee$. Thus $\exp(B) \cdot E$ is a generalized almost complex structure on V for any generalized almost complex structure $E \subseteq (V \oplus V^\vee) \otimes \mathbb{C}$ on V. We will call $\exp(B) \cdot E$ the *B-field transform of E defined by B*. We should note here that these type of transformations are sometimes called gauge-transformations and were introduced with that name into real Dirac geometry [9] in [25]. For an overview of these transformations in the Dirac geometry context, see [3]. Similarly, if $\beta \in \bigwedge^2 V$, then

$$\exp(\beta) := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

then $\exp(\beta) \cdot E$ will be called the *β-field transform of E defined by β*. One can also write these transformations in terms of the orthogonal automorphisms $\mathcal{J}$ of $V \oplus V^\vee$. In this case, the actions of B and β are given by $\mathcal{J} \mapsto \exp(B)\mathcal{J}\exp(-B)$ and $\mathcal{J} \mapsto \exp(\beta)\mathcal{J}\exp(-\beta)$, respectively. We can also describe B-field transforms in terms of local spinor representatives: if a generalized almost complex structure on a real vector bundle V is defined by a pure spinor $\phi \in \bigwedge^\bullet V_{\mathrm{C}}^\vee$, and $B \in \bigwedge^2 V^\vee$ then the B-field transform of this structure corresponds to the pure spinor $\exp(-B) \wedge \phi$ [14,15]. The β-field transform corresponds to the pure spinor $\iota_{\exp(\beta)}\phi$ [12,15].

2.3. *Generalized almost Kähler manifolds*

We will need the notion [12,15] of a generalized almost Kähler structure.

Definition 2.4 (*Gualtieri* [12]). A generalized almost Kähler structure on a manifold M is specified by one of the equivalent sets of data.

(1) A pair $(\mathcal{J}, \mathcal{J}')$ of commuting generalized almost complex structures whose product, $G = -\mathcal{J}\mathcal{J}'$ is positive definite with respect to the standard quadratic form $\langle \bullet, \bullet \rangle$ on $T_M \oplus T_M^\vee$.
(2) A quadruple (g, b, J_+, J_-) consisting of a Riemannian metric g, two-form b, and two almost complex structures J_+ and J_- such that the isomorphisms $\omega_+ = gJ_+ : T_M \to T_M^\vee$ and $\omega_- = gJ_- : T_M \to T_M^\vee$ are anti-symmetric and hence correspond to non-degenerate two-forms.

The two sets of data are related explicitly as follows. The $(+1)$ eigenbundle of G is the graph of $g+b : T_M \to T_M^\vee$. Denote this vector bundle by C_+, and the (-1) eigenbundle (which is the graph of $b-g$) by C_-. Then

$$J_\pm = \pi_{T_M} \circ \mathcal{J} \circ (\pi_{T_M}|_{C_\pm})^{-1}.$$

Conversely, given (g, b, J_+, J_-), one defines

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+ + J_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(J_+^\vee + J_-^\vee) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

and

$$\mathcal{J}' = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+ - J_- & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & -(J_+^\vee - J_-^\vee) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}.$$

Using this same notation we have that

$$G = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}. \tag{2.8}$$

Example 2.5 (*Gualtieri [12]*). Notice that this definition naturally generalizes the linear algebraic data of an Kähler manifold. We will refer to this as the ordinary Kähler case. There is an important family of examples which include the ordinary Kähler as a special case. They come from transforming both the complex and symplectic structures which occur in the ordinary Kähler case by the B-field B.

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ BJ + J^\vee B & -J^\vee \end{pmatrix} \tag{2.9}$$

and

$$\mathcal{J}' = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}, \tag{2.10}$$

where $\omega J = -J^\vee \omega$. The ordinary Kähler case of course comes about from setting B to zero.

3. *T*-duality

Our main goal is to extend the usual T-duality transformation of geometric structures on families of tori in a way that will allow us to incorporate generalized (almost) complex structures.

3.1. T-duality in all directions

In its simplest form, T-duality exchanges geometric data on a torus $\boldsymbol{T} \cong (S^1)^{\times n}$ with geometric data on the dual torus $\boldsymbol{T}^\vee$. For instance if the torus $\boldsymbol{T}$ is a complex manifold, then the dual torus is also naturally a complex manifold. This immediately generalizes to translation invariant (hence integrable) generalized complex structures on $\boldsymbol{T}$.

Indeed, choose a realization of $\boldsymbol{T}$ as a quotient $\boldsymbol{T} = V/\Lambda$ of a real n-dimensional vector space V by a sub-lattice $\mathbb{Z}^n \cong \Lambda \subseteq V$. Then specifying a translation invariant generalized complex structure on $\boldsymbol{T}$ is equivalent to specifying a constant generalized complex structure $\mathcal{J} \in GL(V \oplus V^\vee)$ on the vector space V. Now the dual torus $\boldsymbol{T}^\vee$ has a natural realization as the quotient $\boldsymbol{T}^\vee = V^\vee/\mathrm{Hom}(\Lambda, \mathbb{Z})$. Thus, in order to describe the T-dual generalized complex structure on $\boldsymbol{T}^\vee$ it suffices to produce a constant generalized complex structure on $V^\vee$. This can be done in a simple way: Let $\tau : V \oplus V^\vee \to V^\vee \oplus V$ be the transposition of the two summands. Using the natural identification of $V^{\vee\vee}$ with V, we can also view τ as an isomorphism between $V \oplus V^\vee$ and $V^\vee \oplus V^{\vee\vee}$. We will continue to denote by τ the induced isomorphism $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee \to V_{\mathbb{C}}^\vee \oplus V_{\mathbb{C}}^{\vee\vee} \cong V_{\mathbb{C}}^\vee \oplus V_{\mathbb{C}}$. With this notation one has the following proposition.

Proposition 3.1 (Ben-Bassat and Boyarchenko [2]). *The isomorphism τ induces a bijection between generalized complex structures on V and generalized complex structures on $V^\vee$. If E corresponds to $\mathcal{J} \in \mathrm{Aut}_{\mathbb{R}}(V \oplus V^\vee)$, then $\tau(E)$ corresponds to $\tau \circ \mathcal{J} \circ \tau^{-1}$.*

Remark 3.2. Below, we will see that the transformation of the spinor representatives is a Fourier–Mukai type of transformation. The precise form of this transformation is given in Eq. (6.1). Notice that this proposition applies equally to generalized complex structures on the vector space V and to constant generalized complex structures (which are automatically integrable) on V thought of as a manifold. These in turn give generalized complex structures on tori which are quotients of the vector space.

We also have the following remark from [2].

Remark 3.3. Suppose that E is a generalized complex structure on a real vector space V and E' is the B-field transform of E defined by $B \in \bigwedge^2 V^\vee$. Then, obviously, $\tau(E')$ is the β-field transform of $\tau(E)$, defined by the same $B \in \bigwedge^2 V^\vee$ (but viewed now as a bi-vector on $V^\vee$). Thus, the operation τ interchanges B- and β-field transforms.

The relationship from this last remark was exploited in [17] to produce an interesting conjectural relationship to non-commutative geometry.

3.2. More general T-duality

It has been known for some time that the previous example of T-duality generalizes immediately to a whole family of T-duality transformations. This can be found for example [18] and the references therein. More recently Tang and Weinstein [27] applied this observation to Dirac structures to investigate the group of Morita equivalences of real non-commutative tori.

By analogy with the Tang–Weinstein construction we note that if $V = \bigoplus_{i=1}^{m} V_i$ and $W = \bigoplus_{i=1}^{m} W_i$, where each W_i equals either V_i or $V_i^{\vee}$, then the obvious isomorphism τ from $V \oplus V^{\vee}$ to $W \oplus W^{\vee}$ intertwines the canonical quadratic forms and hence it similarly gives a bijection between generalized complex structures on V with those on W. Notice that these transformations are all real and so there is no problem with the transversality condition. In general, one could also consider as duality transformations, isometries τ, from $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^{\vee}$ to $W_{\mathbb{C}} \oplus W_{\mathbb{C}}^{\vee}$ such that $\tau \circ \mathcal{J} \circ \tau^{-1}$ is a generalized complex structure on W for all (or a family of) generalized complex structures $\mathcal{J}$ on V. A special case of this duality can easily be seen to be the right starting point in generalizing the symplectic/complex correspondence in [23]. To see this, let M be a real manifold with trivial tangent bundle, X a real torus with its normal group structure and V the tangent space to X at the identity, thought of as a trivial bundle on M. Let $\hat{X}$ be the dual torus to X. Then $T_{M\times X} \cong \pi^*(T_M \oplus V)$, and $T_{M\times \hat{X}} \cong \hat{\pi}^*(T_M \oplus V^{\vee})$, so for any isomorphism $L : T_M \to V$ we have that

$$\pi^* \begin{pmatrix} 0 & L \\ -L^{-1} & 0 \end{pmatrix} \tag{3.1}$$

is a complex structure on $M \times X$ and

$$\hat{\pi}^* \begin{pmatrix} 0 & L \\ -L^{\vee} & 0 \end{pmatrix} \tag{3.2}$$

is a symplectic structure on $M \times \hat{X}$. Before pulling back, these structures, thought of as generalized complex structures as in Example 2.1 on $V \oplus T_M$ and $V^{\vee} \oplus T_M$, are related by the obvious map

$$V \oplus T_M \oplus V^{\vee} \oplus T_M^{\vee} \to V^{\vee} \oplus T_M \oplus V \oplus T_M^{\vee}.$$

4. Mirror partners of generalized almost complex structures and associated Dirac structures

In this section we consider a manifold M equipped with a real vector bundle V where the rank of V equals the dimension of M. For any connection ∇ on V we show how to build generalized almost complex structures on $X = \mathrm{tot}(V)$ in terms of data on the base manifold M. We show that there is a bijective correspondence between generalized almost complex structures built in this way on X and generalized almost complex structures of the same type on $\hat{X} = \mathrm{tot}(V^{\vee})$ built using $\nabla^{\vee}$.

Let X be the total space of any vector bundle V over a manifold M. Then we have the exact tangent sequence

$$0 \to \pi^* V \xrightarrow{j} T_X \xrightarrow{\mathrm{d}\pi} \pi^* T_M \to 0 \tag{4.1}$$

A connection on the bundle V is by definition a map of sheaves

$$V \xrightarrow{\nabla} V \otimes T_M^\vee$$

satisfying $\nabla(f\sigma) = \sigma \otimes \mathrm{d}f + f\nabla(\sigma)$ for all local sections f of C_M^∞ and σ of V. We can use any such connection to give a splitting of the above tangent sequence. Namely, let

$$\pi^*\nabla : \pi^*V \to \pi^*V \otimes T_X^\vee$$

be the pullback of ∇ and let S be the tautological global section of π^*V on X. Then $D = (\pi^*\nabla)(S)$ provides a map of vector bundles $\pi^*V \leftarrow T_X$. Now its easy to see that this map is a splitting of (4.1). Indeed, given a local frame $\{e_i\}$ of V over an open set $U \subseteq M$, define smooth functions ξ_i on $\pi^{-1}(U)$ by $\xi_i(a_j e_j(m)) = a_i$ for each m in M. Together with the functions $x_i \circ \pi$, for $\{x_i\}$ coordinates on $U \subseteq M$, these form a coordinate system in $\pi^{-1}(U)$ in which we have $j(e_i) = \partial/\partial x i_i$. In these coordinates we have that on $\pi^{-1}(U)$,

$$S = \xi_i \pi^{-1} e_i$$

and so if we define D by

$$D = (\pi^*\nabla)(S) = \pi^{-1}e_i \otimes \mathrm{d}\xi_i + \xi_i \pi^{-1} e_j \otimes \pi^* A_{ji}, \tag{4.2}$$

where $\nabla e_j = e_j \otimes A_{ji}$ then since $\pi^* A_{ji}$ annihilates the image of j we have that

$$D(j(\pi^{-1}e_k)) = (\pi^{-1}e_i)(\mathrm{d}\xi_i\, j(\pi^{-1}e_k)) = \pi^{-1}e_k$$

and so $D \circ j$ is the identity. We will write this splitting on X as

$$0 \longrightarrow \pi^*V \underset{D}{\overset{j}{\rightleftarrows}} T_X \underset{\alpha}{\overset{d\pi}{\rightleftarrows}} \pi^*T_M \longrightarrow 0,$$

Consider the isomorphism

$$F : T_X \oplus T_X^\vee \to \pi^*V \oplus \pi^*T_M \oplus \pi^*V^\vee \oplus \pi^*T_M^\vee, \qquad F = \begin{pmatrix} D & 0 \\ \mathrm{d}\pi & 0 \\ 0 & j^\vee \\ 0 & \alpha^\vee \end{pmatrix}. \tag{4.3}$$

with inverse

$$F^{-1} : \pi^*V \oplus \pi^*T_M \oplus \pi^*V^\vee \oplus \pi^*T_M^\vee \to T_X \oplus T_X^\vee,$$

$$F^{-1} = \begin{pmatrix} j & \alpha & 0 & 0 \\ 0 & 0 & D^\vee & (\mathrm{d}\pi)^\vee \end{pmatrix}. \tag{4.4}$$

These maps intertwine the obvious quadratic forms and therefore if $\underline{\mathcal{J}}$ is a generalized almost complex structure on $V \oplus T_M$ then $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ is a generalized almost complex structure on X.

Definition 4.1. If ∇ is any connection on V then we define a ∇-*lifted generalized almost complex structure* to be a generalized almost complex structure on $X = \mathrm{tot}(V)$ which can be expressed as $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ where $\underline{\mathcal{J}}$ is a generalized almost complex structure on X and F depends on ∇ as explained above.

Now using the dual connection $\nabla^\vee$, we may split the sequence tangent sequence of $\hat{X}$ as

$$\begin{tikzcd} 0 \arrow[r] & \hat{\pi}^* V^\vee \arrow[r, shift left, "\hat{\jmath}"] & T_{\hat{X}} \arrow[l, shift left, "\hat{D}"] \arrow[r, shift left, "d\hat{\pi}"] & \hat{\pi}^* T_M \arrow[l, shift left, "\hat{\alpha}"] \arrow[r] & 0; \end{tikzcd}$$

Of course we will also need the maps

$$\hat{F} : T_X \oplus T_X^\vee \to \pi^* V^\vee \oplus \hat{\pi}^* T_M \oplus \hat{\pi}^* V \oplus \hat{\pi}^* T_M^\vee, \qquad \hat{F} = \begin{pmatrix} \hat{D} & 0 \\ \mathrm{d}\hat{\pi} & 0 \\ 0 & \hat{\jmath}^\vee \\ 0 & \hat{\alpha}^\vee \end{pmatrix} \tag{4.5}$$

with inverse

$$\hat{F}^{-1} : \pi^* V^\vee \oplus \pi^* T_M \oplus \pi^* V \oplus \hat{\pi}^* T_M^\vee \to T_X \oplus T_X^\vee,$$

$$\hat{F}^{-1} = \begin{pmatrix} \hat{\jmath} & \hat{\alpha} & 0 & 0 \\ 0 & 0 & \hat{D}^\vee & (\mathrm{d}\hat{\pi})^\vee \end{pmatrix}. \tag{4.6}$$

Now if we take any

$$\underline{\mathcal{J}} \in GL(V \oplus T_M \oplus V^\vee \oplus T_M^\vee)$$

we can apply the duality transformation along the fibers to get

$$\hat{\underline{\mathcal{J}}} \in GL(V^\vee \oplus T_M \oplus V \oplus T_M^\vee).$$

Clearly this transformation intertwines the quadratic forms and so $\underline{\mathcal{J}}$ is a generalized almost complex structure on $V \oplus T_M$ if and only if $\hat{\underline{\mathcal{J}}}$ is a generalized almost complex structure on $V^\vee \oplus T_M$. Therefore $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ is a generalized almost complex structure on X if and only $\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^* \hat{\underline{\mathcal{J}}})\hat{F}$ is a generalized almost complex structure on $\hat{X}$. At this point we will impose an extra constraint on these structures.

Definition 4.2. A ∇-lifted generalized almost complex structure $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ will be called *adapted* if

$$\underline{\mathcal{J}}(V \oplus V^\vee) = T_M \oplus T_M^\vee.$$

We will assume that $\mathcal{J}$ is an adapted, ∇-lifted generalized almost complex structure from now on.

Remark 4.3. It is clear from the construction above that $\mathcal{J}$ is adapted if and only if $\hat{\mathcal{J}}$ is.

Finally, let us record the explicit formulas for the operators $\underline{\mathcal{J}}$, $\hat{\underline{\mathcal{J}}}$, $\mathcal{J}$ and $\hat{\mathcal{J}}$. The adapted condition together with the fact that $\underline{\mathcal{J}}^2 = -1$ and that $\underline{\mathcal{J}}$ preserves the quadratic form ensure that it is of the form

$$\underline{\mathcal{J}} = \begin{pmatrix} 0 & \mathcal{J}_{12} & 0 & \mathcal{J}_{22} \\ \mathcal{J}_{13} & 0 & -\mathcal{J}_{22}^{\vee} & 0 \\ 0 & \mathcal{J}_{31} & 0 & -\mathcal{J}_{13}^{\vee} \\ -\mathcal{J}_{31}^{\vee} & 0 & -\mathcal{J}_{12}^{\vee} & 0 \end{pmatrix}, \quad \underline{\mathcal{J}} \in GL(V \oplus T_M \oplus V^{\vee} \oplus T_M^{\vee}) \tag{4.7}$$

subject to

$$\mathcal{J}_{12}\mathcal{J}_{13} - \mathcal{J}_{22}\mathcal{J}_{31}^{\vee} = -1, \tag{4.8}$$

$$\mathcal{J}_{12}\mathcal{J}_{22}^{\vee} + \mathcal{J}_{22}\mathcal{J}_{12}^{\vee} = 0, \tag{4.9}$$

$$\mathcal{J}_{13}\mathcal{J}_{12} - \mathcal{J}_{22}^{\vee}\mathcal{J}_{31} = -1, \tag{4.10}$$

$$\mathcal{J}_{13}\mathcal{J}_{22} + \mathcal{J}_{22}^{\vee}\mathcal{J}_{13}^{\vee} = 0, \tag{4.11}$$

$$\mathcal{J}_{31}\mathcal{J}_{13} + \mathcal{J}_{13}^{\vee}\mathcal{J}_{31}^{\vee} = 0, \tag{4.12}$$

$$\mathcal{J}_{31}^{\vee}\mathcal{J}_{12} + \mathcal{J}_{12}^{\vee}\mathcal{J}_{31} = 0. \tag{4.13}$$

With this notation we have

$$\hat{\underline{\mathcal{J}}} = \begin{pmatrix} 0 & \mathcal{J}_{31} & 0 & -\mathcal{J}_{13}^{\vee} \\ -\mathcal{J}_{22}^{\vee} & 0 & \mathcal{J}_{13} & 0 \\ 0 & \mathcal{J}_{12} & 0 & \mathcal{J}_{22} \\ -\mathcal{J}_{12}^{\vee} & 0 & -\mathcal{J}_{31}^{\vee} & 0 \end{pmatrix}, \quad \hat{\underline{\mathcal{J}}} \in GL(V^{\vee} \oplus T_M \oplus V \oplus T_M^{\vee}) \tag{4.14}$$

and so

$$\mathcal{J} = \begin{pmatrix} j(\pi^*\mathcal{J}_{12})(\mathrm{d}\pi) + \alpha(\pi^*\mathcal{J}_{13})D & j(\pi^*\mathcal{J}_{22})\alpha^{\vee} - \alpha(\pi^*\mathcal{J}_{22}^{\vee})j^{\vee} \\ D^{\vee}(\pi^*\mathcal{J}_{31})(\mathrm{d}\pi) - (\mathrm{d}\pi)^{\vee}(\pi^*\mathcal{J}_{31}^{\vee})D & -D^{\vee}(\pi^*\mathcal{J}_{13}^{\vee})\alpha^{\vee} - (\mathrm{d}\pi)^{\vee}(\pi^*\mathcal{J}_{12}^{\vee})j^{\vee} \end{pmatrix} \tag{4.15}$$

and

$$\hat{\mathcal{J}} = \begin{pmatrix} \hat{j}(\hat{\pi}^*\mathcal{J}_{31})(\mathrm{d}\hat{\pi}) - \hat{\alpha}(\hat{\pi}^*\mathcal{J}_{22}^{\vee})\hat{D} & -\hat{j}(\hat{\pi}^*\mathcal{J}_{13}^{\vee})\hat{\alpha}^{\vee} + \hat{\alpha}(\hat{\pi}^*\mathcal{J}_{13})\hat{j}^{\vee} \\ \hat{D}^{\vee}(\hat{\pi}^*\mathcal{J}_{12})(\mathrm{d}\hat{\pi}) - (\mathrm{d}\hat{\pi})^{\vee}(\pi^*\mathcal{J}_{12}^{\vee})\hat{D} & \hat{D}^{\vee}(\hat{\pi}^*\mathcal{J}_{22})\hat{\alpha}^{\vee} - (\mathrm{d}\hat{\pi})^{\vee}(\hat{\pi}^*\mathcal{J}_{31}^{\vee})\hat{j}^{\vee} \end{pmatrix}. \tag{4.16}$$

Remark 4.4. Notice that the mirror symmetry transformation "exchanges" $\mathcal{J}_{12}$ with $\mathcal{J}_{31}$ and $\mathcal{J}_{22}$ with $-\mathcal{J}_{13}^{\vee}$.

We have written down the bijective correspondence between ∇-lifted, adapted, generalized almost complex structures on X and $\nabla^{\vee}$-lifted, adapted, generalized almost complex structures on $\hat{X}$. We will show below, in the case that ∇ is flat, that $\mathcal{J}$ is integrable if and only if $\hat{\mathcal{J}}$ is.

4.1. Associated almost Dirac structures

For each of the generalized complex structures on X that we consider, there is a natural almost Dirac structure that appears on the base manifold M. It does not depend on the connection used to split the tangent sequence of $X \to M$. An almost Dirac structure on M is just [9] a maximally isotropic sub-bundle of $T_M \oplus T_M^{\vee}$. Now the isomorphism $\underline{\mathcal{J}}$, given in Eq. (4.7), preserves the quadratic form and when restricted to $V \oplus V^{\vee}$, gives an isomorphism $V \oplus V^{\vee} \to T_M \oplus T_M^{\vee}$ which preserves the obvious quadratic forms. Hence the image of V is a maximally isotropic subspace of $T_M \oplus T_M^{\vee}$. In other words it is an almost Dirac structure on M which we will call Δ, where

$$\Delta = \underline{\mathcal{J}}(V) = \mathcal{J}_{13}(V) - \mathcal{J}_{31}^{\vee}(V) = \underline{\hat{\mathcal{J}}}(V).$$

Example 4.5.

(1) Suppose we use our method to construct an almost complex structure on $X = \mathrm{tot}(V)$ out of some arbitrary connection on V. Then we necessarily have that $\mathcal{J}_{13}$ is an isomorphism and $\mathcal{J}_{31} = 0$. Hence $\Delta = T_M$.
(2) If instead we put an almost symplectic structure on $X = \mathrm{tot}(V)$ then $\Delta = T_M^{\vee}$.

Notice that the almost Dirac structure

$$\hat{\Delta} = \underline{\hat{\mathcal{J}}}(V^{\vee}) = -\mathcal{J}_{22}(V^{\vee}) - \mathcal{J}_{12}^{\vee}(V^{\vee}) = \underline{\mathcal{J}}(V^{\vee})$$

arising from the mirror generalized almost complex structure is always transverse to Δ. Hence we always get a pair

$$\Delta \oplus \hat{\Delta} = T_M \oplus T_M^{\vee}$$

of complementary almost Dirac structures. Later we will return to these structures and study their integrability and the existence of flat connections on them.

4.2. Mirror symmetry for generalized almost Kähler manifolds

In this section, we study the case of a pair of ∇-lifted, adapted generalized almost complex structures on the total space of a vector bundle which form a generalized almost Kähler structure as described in Section 2.3. Under these conditions, we write down the mirror transformation rule that allows us to relate the generalized almost Kähler metric G on X and the mirror generalized almost Kähler metric $\hat{G}$ on $\hat{X}$. We observe that in general, the local transformation rules for the pair (g, b) that exist in the physics literature, continue to hold in this setting, even though here neither $\mathcal{J}$ nor $\mathcal{J}'$ needs to be a B-field transform of a generalized complex structure of complex type.

First of all notice that the mirror transform of a generalized almost Kähler pair $(\mathcal{J}, \mathcal{J}')$ is also generalized almost Kähler. Indeed, if we let $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ and $\mathcal{J}' = F^{-1}(\pi^* \underline{\mathcal{J}}')F$, then $\mathcal{J}$ and $\mathcal{J}'$ commute if and only if $\underline{\mathcal{J}}$ and $\underline{\mathcal{J}}'$ commute. This, in turn, is equivalent to $\underline{\hat{\mathcal{J}}}$ and $\underline{\hat{\mathcal{J}}}'$ commuting which happens if and only if $\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^* \underline{\hat{\mathcal{J}}})\hat{F}$ and $\hat{\mathcal{J}}' = \hat{F}^{-1}(\hat{\pi}^* \underline{\hat{\mathcal{J}}}')\hat{F}$ commute. Similarly, $G = -F^{-1}\pi^*(\underline{\mathcal{J}\mathcal{J}'})F$ is positive definite if and only if $-\underline{\mathcal{J}\mathcal{J}'}$ is. This is equivalent to $-\underline{\hat{\mathcal{J}}\hat{\mathcal{J}}'}$ being positive definite, which happens if and only if $\hat{G} = -\hat{F}^{-1}\hat{\pi}^*(\underline{\hat{\mathcal{J}}\hat{\mathcal{J}}'})\hat{F}$ is positive definite. By our assumptions on $\mathcal{J}$ and $\mathcal{J}'$ we may write $\underline{G} = -\underline{\mathcal{J}\mathcal{J}'}$ as

$$G : V \oplus T_M \oplus V^\vee \oplus T_M^\vee \to V \oplus T_M \oplus V^\vee \oplus T_M^\vee,$$

$$\underline{G} = -\underline{\mathcal{J}\mathcal{J}'} = \begin{pmatrix} G_{11} & 0 & G_{21} & 0 \\ 0 & G_{14} & 0 & G_{24} \\ G_{31} & 0 & G_{11}^\vee & 0 \\ 0 & G_{34} & 0 & G_{14}^\vee \end{pmatrix}, \tag{4.17}$$

where $G_{21} = G_{21}^\vee$, $G_{24} = G_{24}^\vee$, $G_{31} = G_{31}^\vee$, $G_{34} = G_{34}^\vee$. Finally, using the fact that this matrix squares to the identity, we get:

$$\begin{aligned} G_{11} &= -\mathcal{J}_{12}\mathcal{J}'_{12} + \mathcal{J}_{22}\mathcal{J}'^{\vee}_{31}, & G_{21} &= \mathcal{J}_{12}\mathcal{J}'^{\vee}_{21} + \mathcal{J}_{22}\mathcal{J}'^{\vee}_{11}, \\ G_{14} &= -\mathcal{J}_{13}\mathcal{J}'^{\vee}_{11} + \mathcal{J}^\vee_{22}\mathcal{J}'_{31}, & G_{24} &= -\mathcal{J}_{13}\mathcal{J}'_{21} - \mathcal{J}^\vee_{22}\mathcal{J}'^{\vee}_{12}, \\ G_{31} &= -\mathcal{J}_{31}\mathcal{J}'_{12} - \mathcal{J}^\vee_{13}\mathcal{J}'^{\vee}_{31}, & G_{34} &= \mathcal{J}^\vee_{31}\mathcal{J}'_{11} + \mathcal{J}^\vee_{12}\mathcal{J}'_{31}. \end{aligned}$$

Therefore $\underline{G}' = -\underline{\hat{\mathcal{J}}\hat{\mathcal{J}}'}$ comes out to be

$$\underline{G}' = -\underline{\hat{\mathcal{J}}\hat{\mathcal{J}}'} = \begin{pmatrix} G_{11}^\vee & 0 & G_{31} & 0 \\ 0 & G_{14} & 0 & G_{24} \\ G_{21} & 0 & G_{11} & 0 \\ 0 & G_{34} & 0 & G_{14}^\vee \end{pmatrix}. \tag{4.18}$$

Remark 4.6. Notice that the mirror symmetry transformation "exchanges" G_{11} with $G_{11}^\vee$, G_{21} with G_{31}, and "preserves" G_{14} and G_{34}.

Now writing G in terms of g and b [12]:

$$G = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} \tag{4.19}$$

and similarly writing $\hat{G}$ in terms of $\hat{g}$ and $\hat{b}$ we can easily manipulate the resulting equations to yield the following formulas for the metrics and B-fields in terms of the vector bundle maps G_{ij} on the base manifold.

$$g = D^\vee \pi^* G_{21}^{-1} D + (\mathrm{d}\pi)^\vee \pi^* G_{24}^{-1}\, \mathrm{d}\pi,$$
$$b = D^\vee \pi^* (G_{11}^\vee G_{21}^{-1}) D + (\mathrm{d}\pi)^\vee \pi^* (G_{14}^\vee G_{24}^{-1})\, \mathrm{d}\pi,$$
$$\hat{g} = \hat{D}^\vee \hat{\pi}^* G_{31}^{-1} \hat{D} + (\mathrm{d}\hat{\pi})^\vee \hat{\pi}^* G_{24}^{-1}\, \mathrm{d}\hat{\pi},$$
$$\hat{b} = \hat{D}^\vee \hat{\pi}^* (G_{11} G_{31}^{-1}) \hat{D} + (\mathrm{d}\hat{\pi})^\vee \hat{\pi}^* (G_{14}^\vee G_{24}^{-1})\, \mathrm{d}\hat{\pi}.$$

Notice that our assumptions on the compatibility of the generalized complex structures, and the foliation and transverse vector bundle, imply that the metric g and B-field b do not mix the horizontal and vertical directions.

Now if we chose local vertical coordinates adapted to the flat connection, y^α on X and $\hat{y}^\alpha$ on $\hat{X}$ and x^i on the base then the above just means that locally we have

$$g = g_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j + h_{\alpha\beta}(x)\, \mathrm{d}y^\alpha\, \mathrm{d}y^\beta, \qquad b = b_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j + B_{\alpha\beta}(x)\, \mathrm{d}y^\alpha\, \mathrm{d}y^\beta,$$
$$\hat{g} = g_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j + \hat{h}_{\alpha\beta}(x)\, \mathrm{d}\hat{y}^\alpha\, \mathrm{d}\hat{y}^\beta, \qquad \hat{b} = b_{ij}(x)\mathrm{d}x^i\, \mathrm{d}x^j + \hat{B}_{\alpha\beta}(x)\, \mathrm{d}\hat{y}^\alpha\, \mathrm{d}\hat{y}^\beta,$$

where of course, x^i means $x^i \circ \pi$ on X and $x^i \circ \hat{\pi}$ on $\hat{X}$.

Then the Buscher transformation rules [5,6] (we used [10] as a reference)

$$(h + B)\hat{h}(h - B) = h \quad \text{and} \quad (h + B)\hat{B}(h - B) = -B$$

are verified from the easily checked identities

$$(G_{21}^{-1} + G_{11}^\vee G_{21}^{-1}) G_{31}^{-1} (G_{21}^{-1} - G_{11}^\vee G_{21}^{-1}) = G_{21}^{-1}$$

and

$$(G_{21}^{-1} + G_{11}^\vee G_{21}^{-1}) G_{11} {G_{31}}^{-1} (G_{21}^{-1} - G_{11}^\vee G_{21}^{-1}) = -G_{11}^\vee {G_{21}}^{-1}$$

respectively.

We now work out the transformation rules relating the two almost complex structures, J_+, J_-, and their mirror partners $\hat{J}_+$ and $\hat{J}_-$. We have

$$J_+ = \mathcal{J}_1 + \mathcal{J}_2(g + b) \quad \text{and} \quad J_- = \mathcal{J}_1 + \mathcal{J}_2(b - g).$$

By combining the results above we can easily compute that

$$J_+ = j(\pi^*(\mathcal{J}_{12} + \mathcal{J}_{22}(G_{14}^\vee + 1)G_{24}^{-1}))\, \mathrm{d}\pi + \alpha(\pi^*(\mathcal{J}_{13} - \mathcal{J}_{22}^\vee(G_{11}^\vee + 1)G_{21}^{-1}))D$$

and

$$J_- = j(\pi^*(\mathcal{J}_{12} + \mathcal{J}_{22}(G_{14}^\vee - 1)G_{24}^{-1}))\, \mathrm{d}\pi + \alpha(\pi^*(\mathcal{J}_{13} - \mathcal{J}_{22}^\vee(G_{11}^\vee - 1)G_{21}^{-1}))D.$$

Hence

$$\hat{J}_{+}=\hat{j}(\hat{\pi}^{*}(\mathcal{J}_{31}-\mathcal{J}_{13}^{\vee}(G_{14}^{\vee}+1)G_{24}^{-1}))\,d\hat{\pi}+\hat{\alpha}(\hat{\pi}^{*}(-\mathcal{J}_{22}^{\vee}+\mathcal{J}_{13}(G_{11}+1)G_{31}^{-1}))\hat{D}$$

and

$$\hat{J}_{-}=\hat{j}(\hat{\pi}^{*}(\mathcal{J}_{31}-\mathcal{J}_{13}^{\vee}(G_{14}^{\vee}-1)G_{24}^{-1}))\,d\hat{\pi}+\hat{\alpha}(\hat{\pi}^{*}(-\mathcal{J}_{22}^{\vee}+\mathcal{J}_{13}(G_{11}-1)G_{31}^{-1}))\hat{D}.$$

5. Branes

In this section, we give some ideas of how one can transfer branes [12,17] from a generalized almost complex manifold to its mirror partner. We will present in detail only a very restricted case. This construction closely parallels that in [20,22]. Consider the following definition from [17] which also appears in a more general form in [12].

Definition 5.1 (*Kapustin* [17]). Let $(X, \mathcal{J})$ be a generalized (almost) complex manifold. Consider triples

$$(Y, \mathcal{L}, \nabla_{\mathcal{L}}),$$

where $f : Y \hookrightarrow X$ is a sub-manifold of X, $\mathcal{L}$ is a Hermitian line bundle on Y, and $\nabla_{\mathcal{L}}$ is a connection on $\mathcal{L}$. Such a triple is said to be a *generalized complex brane* if the bundle

$$\{(v, \alpha) \in T_Y \oplus (T_X^{\vee}|_Y) | f^{*} \circ (\mathrm{d}f)^{\vee}\alpha = \iota_v \mathcal{F}\}$$

is preserved by the restriction of $\mathcal{J}$ to Y, where $\mathcal{F}$ is the curvature two-form of $\nabla_{\mathcal{L}}$.

We studied some special cases [12] of these branes in [2] under the name of generalized Lagrangian sub-manifolds and found some interesting relationships to sub-manifolds of X which inherit generalized complex structures (which we call generalized complex sub-manifolds).

Suppose that M is an n-manifold, V is a rank n vector bundle on M, ∇ is a connection on V, X is the total space of V, $\hat{X}$ is the total space of $V^{\vee}$ and $\mathcal{J}$ is an adapted, ∇-lifted (see Section 4) generalized almost complex structure on V. Let S be a sub-manifold of M, $W \subseteq V|_S$ a sub-bundle, Y the total space of W, and $\hat{Y}$ the total space of the sub-bundle $\mathrm{Ann}(W) \subseteq V^{\vee}|_S$. Then we propose that the relationship between Y and $\hat{Y}$ is a special case of a potential generalization of the relationship between A-cycles and B-cycles in mirror symmetry (see e.g. [20] and references therein). We justify this with the following lemma.

Lemma 5.2. *Under the conditions of the preceding paragraph, the triple* $(Y, \mathbb{C} \otimes C_Y^{\infty}, d)$ *is a generalized complex brane of* $(X, \mathcal{J})$ *if and only if the triple* $(\hat{Y}, \mathbb{C} \otimes C_{\hat{Y}}^{\infty}, d)$ *is a generalized complex brane of* $(\hat{X}, \hat{\mathcal{J}})$.

Proof. In this proof, we will be using the notation of Section 4. We need to show that

$$\mathcal{J}(T_Y \oplus \mathrm{Ann}(T_Y)) = T_Y \oplus \mathrm{Ann}(T_Y) \tag{5.1}$$

if and only if

$$\hat{\mathcal{J}}(T_{\hat{Y}} \oplus \mathrm{Ann}(T_{\hat{Y}})) = T_{\hat{Y}} \oplus \mathrm{Ann}(T_{\hat{Y}}), \tag{5.2}$$

where it is to be understood that we are restricting $\mathcal{J}$ to Y and $\hat{\mathcal{J}}$ to $\hat{Y}$. Observe that when understood as bundles on Y, we have

$$T_Y = j(\pi^* W) \oplus \alpha(\pi^* T_S),$$

$$\mathrm{Ann}(T_Y) = D^\vee(\pi^*(\mathrm{Ann}(W))) \oplus (\mathrm{d}\pi)^\vee(\pi^*(\mathrm{Ann}(T_S)))$$

and when understood as bundles on $\hat{Y}$, we have

$$T_{\hat{Y}} = \hat{j}(\hat{\pi}^*(\mathrm{Ann}(W))) \oplus \hat{\alpha}(\hat{\pi}^* T_S), \quad \mathrm{Ann}(T_Y) = \hat{D}^\vee(\hat{\pi}^*(W)) \oplus (\mathrm{d}\hat{\pi})^\vee(\hat{\pi}^*(\mathrm{Ann}(T_S))).$$

From this perspective it is clear that both (5.1) and (5.2) are both equivalent simply to the conditions (understanding that $\underline{\mathcal{J}}$ is restricted to S)

$$\mathcal{J}_{13}(W) \subseteq T_S, \tag{5.3}$$

$$\mathcal{J}_{12}(T_S) \subseteq W, \tag{5.4}$$

$$\mathcal{J}_{22}(\mathrm{Ann}(T_S)) \subseteq W, \tag{5.5}$$

$$\mathcal{J}_{31}(T_S) \subseteq \mathrm{Ann}(W), \tag{5.6}$$

and therefore we are done. A more general treatment will involve replacing W by an affine sub-bundle which will result in a non-trivial line bundle on the mirror side. A more general story will be the result of upcoming work. The extension of the results in this section to the case of torus bundles is clear from the development in Section 3 of Part II of this paper [1]. We hope that in a suitable extended version of the homological mirror symmetry conjecture [19], generalized complex manifolds would be assigned categories in a natural way, and branes would be related to objects in these categories. □

Remark 5.3. On a torus bundle $Z \to M$, where Z is an orientable compact manifold, it is plausible that the correspondence which we are describing here, when thought of as a correspondence between homology classes on Z to homology classes on the dual torus bundle $\hat{Z} \to M$ agrees, upon using Poincaré Duality, with the correspondence in cohomology given in Section 3 of Part II of this paper [1].

6. The mirror transformation on spinors and the Fourier transform

In this section we study a map from certain complex valued differential forms on the total space of a vector bundle to complex valued differential forms on the total space of the dual vector bundle. We show that the line sub-bundle of the bundle of differential forms associated to an adapted, ∇-lifted generalized almost complex structure has a sub-sheaf which goes under this correspondence to the sub-sheaf associated to the mirror generalized almost complex structure. The idea of using a Fourier transform in the context of T-duality for generalized complex structures has appeared in a slightly different context in both [12] (based on ideas appearing in [21]) and also [10,26] and the references therein.

Consider a vector bundle V of rank n on a manifold M. There is an isomorphism

$$\bigwedge V^\vee \otimes \mathbb{C} \to \left(\bigwedge V \otimes \bigwedge^n V^\vee\right) \otimes \mathbb{C}$$

given by

$$\phi \mapsto \int (\phi \wedge \exp(\kappa)), \tag{6.1}$$

where κ is the canonical global section of $(V \otimes V^\vee) \otimes \mathbb{C} \subseteq \bigwedge^2(V \oplus V^\vee) \otimes \mathbb{C}$ and

$$\int : \bigwedge(V \oplus V^\vee) \otimes \mathbb{C} \to \bigwedge V \otimes \bigwedge^n V^\vee \otimes \mathbb{C}$$

is the projection map. Furthermore, this map decomposes into isomorphisms

$$\bigwedge^p V^\vee \otimes \mathbb{C} \to \bigwedge^{n-p} V \otimes \bigwedge^n V^\vee \otimes \mathbb{C},$$

and also induces a Fourier transform isomorphism, which we will call $\mathbf{F.T.}$

$$\bigwedge V^\vee \otimes \bigwedge T_M^\vee \otimes \mathbb{C} \xrightarrow{\mathbf{F.T.}} \bigwedge V \otimes \bigwedge T_M^\vee \otimes \bigwedge^n V^\vee \otimes \mathbb{C}$$

which in turn decompose into isomorphisms

$$\left(\bigwedge^q T_M^\vee \otimes \bigwedge^p V^\vee \otimes \mathbb{C}\right) \to \left(\bigwedge^q T_M^\vee \otimes \bigwedge^{n-p} V \otimes \bigwedge^n V^\vee \otimes \mathbb{C}\right).$$

Lemma 6.1. *Let V be a rank n orientable vector bundle on an n-manifold, $\underline{\mathcal{J}}$ a generalized almost complex structure on the vector bundle $V \oplus T_M$ satisfying $\underline{\mathcal{J}}(V \oplus V^\vee) = T_M \oplus T_M^\vee$, and $\hat{\underline{\mathcal{J}}}$ the mirror structure. Then the composition of the map* $\mathbf{F.T.}$ *with any trivialization of $\bigwedge^n V^\vee$ takes the line bundle $\underline{L} \subseteq \bigwedge(V \oplus T_M)^\vee \otimes \mathbb{C}$ which represents $\underline{\mathcal{J}}$ to the line bundle* $\hat{\underline{L}} \subseteq \bigwedge(V \oplus T_M)^\vee \otimes \mathbb{C}$ which represents $\hat{\underline{\mathcal{J}}}$.

Proof. First of all notice that changing the trivialization of $\bigwedge^n V$ multiplies the image of $\mathbf{F.T.}$ by a non-zero function on the base manifold. This is an automorphism of image of the composed map along with its inclusion into $\bigwedge(V \oplus T_M)^\vee \otimes \mathbb{C}$. The $(+i)$ eigenbundle of $\underline{\mathcal{J}}$

is the graph of the map

$$-i\underline{\mathcal{J}}|_{(V\oplus V^\vee)\otimes\mathbb{C}} : (V \oplus V^\vee)\otimes\mathbb{C} \to (T_M \oplus T_M^\vee)\otimes\mathbb{C}.$$

Similarly, the $(+i)$ eigenbundle of $\hat{\underline{\mathcal{J}}}$ is the graph of

$$-i\underline{\mathcal{J}}|_{(V^\vee\oplus V)\otimes\mathbb{C}} : (V \oplus V^\vee)\otimes\mathbb{C} \to (T_M \oplus T_M^\vee)\otimes\mathbb{C}.$$

The sections ϕ of L therefore satisfy

$$\iota_{v-i\mathcal{J}_{13}v}\phi + i(\mathcal{J}_{31}^\vee v)\bigwedge \phi = 0, \tag{6.2}$$

$$\iota_{i\mathcal{J}_{22}^\vee\alpha}\phi + (\alpha + i\mathcal{J}_{12}^\vee\alpha)\bigwedge \phi = 0 \tag{6.3}$$

for all sections v of V and α of $V^\vee$.

We need to show that the section $\mathbf{F.T.}(\phi)$ of $\bigwedge(V^\vee \oplus T_M)^\vee \otimes \bigwedge^n V^\vee \otimes \mathbb{C}$ satisfies

$$\iota_{-i\mathcal{J}_{13}v}\mathbf{F.T.}(\phi) + (v + i\mathcal{J}_{31}^\vee v)\bigwedge \mathbf{F.T.}(\phi) = 0, \tag{6.4}$$

$$\iota_{\alpha+i\mathcal{J}_{22}^\vee\alpha}\mathbf{F.T.}(\phi) + (i\mathcal{J}_{12}^\vee\alpha)\bigwedge \mathbf{F.T.}(\phi) = 0 \tag{6.5}$$

for all sections v of V and α of $V^\vee$. These equations hold for the map $\mathbf{F.T.}$ if and only if they hold for the composition of $\mathbf{F.T.}$ with any trivialization. This can be seen by writing $\mathbf{F.T.}$ as the composed map followed by the action of “wedging” with a global section of $\bigwedge^n V^\vee$. Eqs. (6.4) and (6.5) will follow immediately from taking the Fourier transform of both sides of (6.2) and (6.3) and using the following lemma. $\square$

Lemma 6.2. *For any sections ζ of $\bigwedge(V \oplus T_M)^\vee \otimes \mathbb{C}$, v of $V \otimes \mathbb{C}$, w of $T_M \otimes \mathbb{C}$, α of $V^\vee \otimes \mathbb{C}$ and β of $T_M^\vee \otimes \mathbb{C}$ we have*

(i) $\mathbf{F.T.}(\iota_v\zeta) = v \bigwedge \mathbf{F.T.}(\zeta)$;
(ii) $\mathbf{F.T.}(\iota_w\zeta) = \iota_w\mathbf{F.T.}(\zeta)$;
(iii) $\mathbf{F.T.}(\alpha \bigwedge \zeta) = \iota_\alpha\mathbf{F.T.}(\zeta)$;
(iv) $\mathbf{F.T.}(\beta \bigwedge \zeta) = \beta \bigwedge \mathbf{F.T.}(\zeta)$.

Proof. It clearly suffices to prove this in the case that ζ is a section of $\bigwedge^p(V \oplus T_M)^\vee$. Notice also that $\int \circ\, \iota_v = 0$ and $\iota_v\kappa = -v$ for any section v of V and $\iota_\alpha\kappa = \alpha$ for any section α of $V^\vee$. Then we have

$$\begin{aligned}\mathbf{F.T.}(\iota_v\zeta) &= \int (\iota_v\zeta)\bigwedge \exp(\kappa) = \int \iota_v(\zeta \bigwedge \exp(\kappa)) - (-1)^p \int \zeta \bigwedge \iota_v \exp(\kappa)\\ &= -(-1)^p \int \zeta \bigwedge \iota_v \exp(\kappa) = -(-1)^p \int \zeta \bigwedge (\iota_v(\kappa)) \bigwedge \exp(\kappa)\\ &= (-1)^p \int \zeta \bigwedge v \bigwedge \exp(\kappa) = \int v \bigwedge \zeta \bigwedge \exp(\kappa)\end{aligned}$$

$$= v \bigwedge \int \zeta \bigwedge \exp(\kappa) = v \bigwedge \mathbf{F}.\mathbf{T}.(\zeta),$$

$$\mathbf{F}.\mathbf{T}.(\iota_w \zeta) = \int (\iota_w \zeta) \bigwedge \exp(\kappa) = \int \iota_w(\zeta \bigwedge \exp(\kappa))$$

$$= \iota_w \int (\zeta \bigwedge \exp(\kappa)) = \iota_w \mathbf{F}.\mathbf{T}.(\zeta),$$

$$\mathbf{F}.\mathbf{T}.(\alpha \bigwedge \zeta) = \int (\alpha \bigwedge \zeta \bigwedge \exp(\kappa)) = (-1)^p \int (\zeta \bigwedge \alpha \bigwedge \exp(\kappa))$$

$$= (-1)^p \int (\zeta \bigwedge \iota_\alpha \exp(\kappa)) = \iota_\alpha \int (\zeta \bigwedge \exp(\kappa)) = \iota_\alpha \mathbf{F}.\mathbf{T}.(\zeta),$$

$$\mathbf{F}.\mathbf{T}.(\beta \bigwedge \zeta) = \int \beta \bigwedge \zeta \bigwedge \exp(\kappa) = \beta \bigwedge \int \zeta \bigwedge \exp(\kappa) = \beta \bigwedge \mathbf{F}.\mathbf{T}.(\zeta) \qquad \square$$

Let M an n-manifold and $X \to^{\pi} M$ be the total space of an orientable vector bundle V on M, with connection ∇ and $\mathcal{J}$ a ∇-lifted, adapted generalized almost complex structure on X. Let $\hat{X} \to^{\hat{\pi}} M$ be the total space of $V^\vee$. Using ∇ we may realize $\pi^*(\bigwedge V^\vee \otimes \bigwedge T_M^\vee \otimes \mathbb{C})$ as a sub-bundle of $\bigwedge T_X^\vee \otimes \mathbb{C}$. Now $\mathcal{J}$ determines a spinorial line bundle $L \subseteq \bigwedge T_X^\vee \otimes \mathbb{C}$ which is simply the image under this isomorphism of the pullback $\pi^* \underline{L}$. Similarly, $\hat{\mathcal{J}}$ determines a spinorial line bundle $\hat{L} \subseteq \bigwedge T_{\hat{X}}^\vee \otimes \mathbb{C}$ isomorphic to $\hat{\pi}^* \underline{\hat{L}}$. Therefore, interpreting the Fourier transform maps as isomorphisms $\pi_* \pi^{-1} \underline{L} \to \hat{\pi}_* \hat{\pi}^{-1} \underline{\hat{L}}$ we can map certain sections of L over open sets of the form $\pi^{-1}(U)$ to sections of $\hat{L}$ over open sets of the form $\hat{\pi}^{-1}(U)$. We have shown the following lemma.

Lemma 6.3. *Let V is an orientable rank n vector bundle on an n-manifold M and $\mathcal{J}$ an adapted, ∇-lifted generalized almost complex structure on $X = \mathrm{tot}(V)$ with associated line bundle L. Let the mirror generalized almost complex structure have associated line bundle $\hat{L}$. Then their are sub-sheaves, $\pi^{-1}\underline{L} \subseteq L$ and $\hat{\pi}^{-1}\underline{\hat{L}} \subseteq \hat{L}$ such that if we compose the isomorphism*

$$\pi_* \pi^{-1} \left(\bigwedge T_M^\vee \otimes \bigwedge V^\vee \otimes \mathbb{C} \right) \stackrel{\mathbf{F.T.}}{\to} \hat{\pi}_* \hat{\pi}^{-1} \left(\bigwedge T_M^\vee \otimes \bigwedge V \otimes \bigwedge^n V^\vee \otimes \mathbb{C} \right)$$

with any trivialization of $\bigwedge^n V^\vee$, the resulting isomorphism

$$\pi_* \left(\bigwedge T_X^\vee \otimes \mathbb{C} \right) \supseteq \pi_* \pi^{-1} \left(\bigwedge T_M^\vee \otimes \bigwedge V^\vee \otimes \mathbb{C} \right)$$
$$\to \hat{\pi}_* \hat{\pi}^{-1} \left(\bigwedge T_M^\vee \otimes \bigwedge V \otimes \mathbb{C} \right) \subseteq \hat{\pi}_* \left(\bigwedge T_{\hat{X}}^\vee \otimes \mathbb{C} \right)$$

restricts to an isomorphism

$$\pi_* L \supseteq \pi_* \pi^{-1} \underline{L} \to \hat{\pi}_* \hat{\pi}^{-1} \underline{\hat{L}} \subseteq \hat{\pi}_* \hat{L}.$$

This is useful because, from the ∇-lifted property, its easy to see that $L = \pi^{-1}\underline{L} \otimes C^{\infty}_{X}$ and $\hat{L} = \hat{\pi}^{-1}\underline{\hat{L}} \otimes C^{\infty}_{\hat{X}}$. Therefore for U small enough, representative spinors for $\mathcal{J}$ over $\pi^{-1}(U)$ and $\hat{\mathcal{J}}$ over $\hat{\pi}^{-1}(U)$ exist and can be chosen as pullbacks of sections of $\underline{L}$ and $\underline{\hat{L}}$ over U. They are exchanged under the Fourier transform even though we have not written down a map between the push-forwards of L and $\hat{L}$. The situation will be much more simple in the case of torus bundles.

Remark 6.4. It is important to remember that the geometry of $\mathcal{J}$ is not just captured by the abstract line bundle L up to isomorphism, but rather, by L together with its embedding into the differential forms.

Understanding mirror symmetry in terms of a relationship between pure spinors was approached with similar techniques in [10].

7. Transverse foliations and generalized Kähler geometry

In this section we study in the abstract some of essential geometric details of our construction without reference to the specific context (e.g. the type of bundle).

Definition 7.1. Suppose that X is a foliated manifold and let $\mathcal{P} \subseteq T_X$ be the involute sub-bundle tangent to the leaves of the foliation. We say that a generalized complex structure $\mathcal{J}$ and $\mathcal{P}$ are *compatible* if there exists a complementary sub-bundle $\mathcal{Q} \subseteq T_X$ so that

$$\mathcal{J}_{|\mathcal{P}\oplus \mathrm{Ann}(\mathcal{Q})} : \mathcal{P} \oplus \mathrm{Ann}(\mathcal{Q}) \to \mathcal{Q} \oplus \mathrm{Ann}(\mathcal{P})$$

is an isomorphism of vector bundles. Under this condition, we will call $\mathcal{Q}$ a $\mathcal{J}$-compliment to $\mathcal{P}$. For $\mathcal{Q}$ a $\mathcal{J}$-compliment to $\mathcal{P}$ we will often tacitly identify $\mathcal{P}^{\vee}$ with Ann$\mathcal{Q}$, and $\mathcal{Q}^{\vee}$ with Ann$\mathcal{P}$.

Notice that the $(+i)$ eigenbundle, E of $\mathcal{J}$ is in this case necessarily transverse to both $\mathcal{P}_{\mathrm{C}} \oplus \mathrm{Ann}(\mathcal{Q})_{\mathrm{C}}$ and $\mathcal{Q}_{\mathrm{C}} \oplus \mathrm{Ann}(\mathcal{P})_{\mathrm{C}}$. Hence E is the graph of a map from $\mathcal{P}_{\mathrm{C}} \oplus \mathrm{Ann}(\mathcal{Q})_{\mathrm{C}}$ to $\mathcal{Q}_{\mathrm{C}} \oplus \mathrm{Ann}(\mathcal{P})_{\mathrm{C}}$. In fact, it is easy to see that we have $E = \mathrm{graph}(-i\mathcal{J})$ where we consider $(-i\mathcal{J})$ as a map from $\mathcal{P}_{\mathrm{C}} \oplus \mathrm{Ann}(\mathcal{Q})_{\mathrm{C}}$ to $\mathcal{Q}_{\mathrm{C}} \oplus \mathrm{Ann}(\mathcal{P})_{\mathrm{C}}$.

Example 7.2.

(i) Suppose that X is a manifold equipped with an involute distribution $\mathcal{P} \subseteq T_X$ of half the dimension of X. Let $\mathcal{J}$ be the generalized almost complex structure on X corresponding to a non-degenerate real two-form ω. Then $\mathcal{P}$ and $\mathcal{J}$ are compatible if and only if $\mathcal{P}$ defines a Lagrangian foliation on X. Indeed, the compatibility shows that ω defines an isomorphism from $\mathcal{P}$ to Ann$\mathcal{P}$, which shows that $\mathcal{P}$ is Lagrangian. Conversely, if $\mathcal{P}$ is Lagrangian, then by (see [7]) choosing an almost complex structure J so that the isomorphism $-\omega J : T_X \to T_X^{\vee}$ represents a Riemannian metric on X, it is easy to see that the vector bundle $J\mathcal{P}$ is a $\mathcal{J}$-compliment to $\mathcal{P}$, and so $J\mathcal{P}$ is also Lagrangian. This example signifies some relationship of the content of this paper with the area of integrable systems.

(ii) On the other hand if X is a manifold equipped with an involute distribution $\mathcal{P} \subseteq T_X$ of half the dimension of X and $\mathcal{J}$ is the generalized almost complex structure on X corresponding to an almost complex structure J then $\mathcal{P}$ and $\mathcal{J}$ are compatible if and only if $J\mathcal{P} \cap \mathcal{P} = (0)$. In other words the leaves of the foliation are totally real sub-manifolds [4]. In this case the $\mathcal{J}$-compliment to $\mathcal{P}$ is fixed uniquely as $J\mathcal{P}$.

(iii) One of the main classes of examples in Part II of this paper is where X is an n-torus bundle with section over an n-manifold, $\mathcal{J}$ is a semi-flat generalized complex structure, $\mathcal{P}$ is the vertical foliation tangent to the torus fibers, and $\mathcal{Q}$ is the horizontal foliation given by the splitting of the tangent sequence given by the connection as will be explained in Section 2 of Part II.

Remark 7.3. In the above and in much of what follows, the fact that $\mathcal{P}$ is involute is irrelevant. That is to say, it could just be a sub-bundle of the tangent bundle. However, it will be taken to be involute for the applications that we have in mind, for instance when $\mathcal{P}$ represents the tangent directions to a torus fibration, as will be discussed in Part II [1].

Definition 7.4. Suppose that $\mathcal{J}$ and $\mathcal{J}'$ constitute a generalized almost Kähler pair of generalized almost complex structures and $\mathcal{P} \subseteq T_X$ is a sub-bundle of the tangent bundle of half the dimension. Then we say that $\mathcal{P}$ is compatible with the pair $(\mathcal{J}, \mathcal{J}')$ when, using the notation introduced in Section 2.1

$$\mathcal{J}_2(\mathrm{Ann}(\mathcal{P})) \subseteq \mathcal{P}, \tag{7.1}$$

$$\mathcal{J}_3(\mathcal{P}) \subseteq \mathrm{Ann}(\mathcal{P}), \tag{7.2}$$

$$\mathcal{J}'_2(\mathrm{Ann}(\mathcal{P})) \subseteq \mathcal{P}, \tag{7.3}$$

$$\mathcal{J}'_3(\mathcal{P}), \subseteq \mathrm{Ann}(\mathcal{P}), \tag{7.4}$$

$$\mathcal{J}_1\mathcal{J}'_1(\mathcal{P}) \subseteq \mathcal{P}, \tag{7.5}$$

$$\mathcal{J}'_1\mathcal{J}_1(\mathcal{P}) \subseteq \mathcal{P}, \tag{7.6}$$

$$\mathcal{J}_4\mathcal{J}'_4(\mathrm{Ann}(\mathcal{P})) \subseteq \mathrm{Ann}(\mathcal{P}), \tag{7.7}$$

$$\mathcal{J}'_4\mathcal{J}_4(\mathrm{Ann}(\mathcal{P})) \subseteq \mathrm{Ann}(\mathcal{P}). \tag{7.8}$$

Notice that if there is a sub-bundle $\mathcal{Q}$ which is both a $\mathcal{J}$-compliment and an $\mathcal{J}'$-compliment to $\mathcal{P}$ then $\mathcal{P}$ is compatible with $(\mathcal{J}, \mathcal{J}')$. The converse will be shown below. In the ordinary Kähler case 2.4 the condition that $\mathcal{P}$ is compatible with $(\mathcal{J}, \mathcal{J}')$ simply says that $\mathcal{P}$ is Lagrangian with respect to the symplectic structure. In the B-transformed almost Kähler case where we have an almost Kähler pair (J, ω) the conditions are as follows: $\mathcal{P}$ must be Lagrangian with respect to the symplectic structure ω and also $B(\omega^{-1}B\mathcal{P}, \mathcal{P}) = 0$ and $B(J\mathcal{P}, \mathcal{P}) = 0$.

Theorem 7.5. *If X is a $2n$-dimensional real manifold then a rank n bundle $\mathcal{P} \subseteq T_X$ is compatible with a generalized almost Kähler pair $(\mathcal{J}, \mathcal{J}')$ if and only if there is a sub-bundle $\mathcal{Q} \subseteq T_X$ which is both a $\mathcal{J}$-compliment and a $\mathcal{J}'$-compliment to $\mathcal{P}$. These properties specify $\mathcal{Q}$ uniquely.*

Proof. If such a $\mathcal{Q}$ exists then it is clear that $\mathcal{P}$ and $\mathcal{Q}$ are both compatible with the pair $(\mathcal{J}, \mathcal{J}')$. Furthermore, the property that $\mathcal{Q}$ is a $\mathcal{J}$-compliment and a $\mathcal{J}'$-compliment to $\mathcal{P}$ for a generalized Kähler pair $(\mathcal{J}, \mathcal{J}')$ fixes $\mathcal{Q}$ uniquely. Indeed, if we are in this situation and $G = -\mathcal{J}\mathcal{J}'$ is the generalized Kähler metric then we have that

$$g - bg^{-1}b = G_3 = -\mathcal{J}'_3\mathcal{J}_1 - \mathcal{J}'_4\mathcal{J}_3.$$

Therefore the isomorphism $(g - bg^{-1}b)$ takes $\mathcal{P}$ to $\mathrm{Ann}(\mathcal{Q})$ and so $\mathcal{Q}$ must be the perpendicular to complement of $\mathcal{P}$ with respect to the metric $(g - bg^{-1}b)$. We will now realize $\mathcal{Q}$ explicitly as the image of a different automorphism $K_+ \in GL(T_X)$ of the tangent bundle with itself which becomes an almost complex structure in the when $b = 0$. Therefore the proof will be completed via the following lemma. $\square$

Lemma 7.6. *If $(\mathcal{J}, \mathcal{J}')$ is a generalized almost Kähler pair then the map*

$$K_+ = J_+(1 - g^{-1}b) = \mathcal{J}_1 + \mathcal{J}'_1$$

is an isomorphism of the tangent bundle with itself. If $\mathcal{P}$ is compatible with the pair $(\mathcal{J}, \mathcal{J}')$ then K_+ takes $\mathcal{P}$ to a sub-bundle $\mathcal{Q}$, transversal to $\mathcal{P}$, and we have that

$$\mathcal{J}_{|\mathcal{P}\oplus\mathrm{Ann}(\mathcal{Q})} : \mathcal{P} \oplus \mathrm{Ann}(\mathcal{Q}) \to \mathcal{Q} \oplus \mathrm{Ann}(\mathcal{P})$$

and

$$\mathcal{J}'_{|\mathcal{P}\oplus\mathrm{Ann}(\mathcal{Q})} : \mathcal{P} \oplus \mathrm{Ann}(\mathcal{Q}) \to \mathcal{Q} \oplus \mathrm{Ann}(\mathcal{P})$$

are isomorphisms of vector bundles. In other words, $\mathcal{Q}$ is both a $\mathcal{J}$-compliment and a $\mathcal{J}'$-compliment to $\mathcal{P}$.

Proof. First of all, notice that K_+ is an isomorphism of the vector bundle T_X with itself. Indeed, the vector bundle map $(g - bg^{-1}b)$ from T_X to $T_X^\vee$ corresponds to the metric $g(v, w) + g^{-1}(bv, bw)$ which is positive definite and hence if we consider f to $(g - bg^{-1}b)^{-1}$, we see that

$$K_+(-(g + b)fJ_+) = 1.$$

Now we have

$$\begin{aligned} G_3\mathcal{J}_1 &= -(\mathcal{J}_3\mathcal{J}'_1 + \mathcal{J}_4\mathcal{J}'_3)\mathcal{J}_1 - \mathcal{J}_3\mathcal{J}'_1\mathcal{J}_1 - \mathcal{J}_4\mathcal{J}'_3\mathcal{J}_1 \\ &= -\mathcal{J}_3\mathcal{J}'_1\mathcal{J}_1 - \mathcal{J}_4(\mathcal{J}_3\mathcal{J}'_1 + \mathcal{J}_4\mathcal{J}'_3 - \mathcal{J}'_4\mathcal{J}_3) \end{aligned}$$

$$
\begin{aligned}
&= -\mathcal{J}_3\mathcal{J}'_1\mathcal{J}_1 - \mathcal{J}_4\mathcal{J}_3\mathcal{J}'_1 - (\mathcal{J}_4)^2\mathcal{J}'_3 + \mathcal{J}_4\mathcal{J}'_4\mathcal{J}_3\\
&= -\mathcal{J}_3\mathcal{J}'_1\mathcal{J}_1 - \mathcal{J}_4\mathcal{J}_3\mathcal{J}'_1 - (-1 - \mathcal{J}_3\mathcal{J}_2)\mathcal{J}'_3 + \mathcal{J}_4\mathcal{J}'_4\mathcal{J}_3\\
&= -\mathcal{J}_3\mathcal{J}'_1\mathcal{J}_1 - \mathcal{J}_4\mathcal{J}_3\mathcal{J}'_1 + \mathcal{J}'_3 + \mathcal{J}_3\mathcal{J}_2\mathcal{J}'_3 + \mathcal{J}_4\mathcal{J}'_4\mathcal{J}_3\\
&= -\mathcal{J}_3\mathcal{J}'_1\mathcal{J}_1 + \mathcal{J}_3\mathcal{J}_1\mathcal{J}'_1 + \mathcal{J}'_3 + \mathcal{J}_3\mathcal{J}_2\mathcal{J}'_3 + \mathcal{J}_4\mathcal{J}'_4\mathcal{J}_3.
\end{aligned}
$$

By inspection of the definition of compatibility of $\mathcal{P}$ with the pair $(\mathcal{J}, \mathcal{J}')$ we have that all of these terms send $\mathcal{P}$ into $\mathrm{Ann}(\mathcal{P})$. Thus $G_3\mathcal{J}_1(\mathcal{P}) \subseteq \mathrm{Ann}(\mathcal{P})$. Since the roles of $\mathcal{J}$ and $\mathcal{J}'$ are interchangeable we have,

$$G_3\mathcal{J}'_1 = -\mathcal{J}'_3\mathcal{J}_1\mathcal{J}'_1 + \mathcal{J}'_3\mathcal{J}'_1\mathcal{J}_1 + \mathcal{J}_3 + \mathcal{J}'_3\mathcal{J}'_2\mathcal{J}_3 + \mathcal{J}'_4\mathcal{J}_4\mathcal{J}'_3$$

as well. Hence $G_3\mathcal{J}'_1(\mathcal{P}) \subseteq \mathrm{Ann}(\mathcal{P})$. Therefore the image of $\mathcal{P}$ under the isomorphism $K_+ = \mathcal{J}_1 + \mathcal{J}'_1$ is the perpendicular sub-bundle to $\mathcal{P}$ with respect to the metric G_3.

Now, in order to show the remaining claims, it suffices to define $\mathcal{Q}$ as $K_+(\mathcal{P})$ and show that $\mathcal{J}_1(\mathcal{Q}) \subseteq \mathcal{P}$ and $\mathcal{J}_4(\mathrm{Ann}(\mathcal{P})) \subseteq \mathrm{Ann}(\mathcal{Q})$. Indeed suppose that we have shown this. Note that reversing the roles of $\mathcal{J}$ and $\mathcal{J}'$ does not change $\mathcal{Q}$ and so we get that $\mathcal{J}'_1(\mathcal{Q}) \subseteq \mathcal{P}$ and $\mathcal{J}'_4(\mathrm{Ann}(\mathcal{P})) \subseteq \mathrm{Ann}(\mathcal{Q})$, and therefore $\mathcal{J}(\mathcal{Q} \oplus \mathrm{Ann}(\mathcal{P})) \subseteq \mathcal{P} \oplus \mathrm{Ann}(\mathcal{Q})$ and $\mathcal{J}'(\mathcal{Q} \oplus \mathrm{Ann}(\mathcal{P})) \subseteq \mathcal{P} \oplus \mathrm{Ann}(\mathcal{Q})$, which is enough since $\mathcal{J}$ and $\mathcal{J}'$ are isomorphisms.

Let v be an element of a fiber of $\mathcal{Q}$. We may express it as $(\mathcal{J}_1 + \mathcal{J}'_1)w$ for a unique fiber w of $\mathcal{P}$ over the same point. Then

$$\mathcal{J}_1 v = \mathcal{J}_1{}^2 w + \mathcal{J}_1\mathcal{J}'_1 w = -w - \mathcal{J}_2\mathcal{J}_3 w + \mathcal{J}_1\mathcal{J}'_1 w$$

which is an element of the fiber of $\mathcal{P}$ over the same point.

Let μ be an element of a fiber of $\mathrm{Ann}(\mathcal{P})$. Then, if v is in the fiber of $\mathcal{Q}$ over the same point, we have that $(\mathcal{J}_4\mu)v = -\mu(\mathcal{J}_1 v)$ which is zero by the previous paragraph. Therefore $\mathcal{J}_4\mu$ is in the fiber of $\mathrm{Ann}(\mathcal{Q})$ over the same point. □

The reader may wonder about the possibility of instead taking

$$K_- = \mathcal{J}_1 - \mathcal{J}'_1 = J_-(1 + g^{-1}b).$$

Lemma 7.7. *K_- is an isomorphism of the tangent bundle with itself. In general it is not equal to K_+. However, if $\mathcal{P}$ is compatible with the generalized almost Kähler pair $(\mathcal{J}, \mathcal{J}')$, we have that $K_+(\mathcal{P}) = K_-(\mathcal{P})$. In fact we have that they are both equal to the orthogonal complement of $\mathcal{P}$ with respect to the metric $G_3 = g - bg^{-1}b$.*

Proof. To see that K_- is an isomorphism, simply note that $-f(g - b)J_-K_- = 1$ where f is the inverse to $g - bg^{-1}b$. Define $\mathcal{Q}_+ = K_+(\mathcal{P})$ and $\mathcal{Q}_- = K_-(\mathcal{P})$. By the above arguments it is clear that $K_- = \mathcal{J}_1 - \mathcal{J}'_1$ is an isomorphism from $\mathcal{P}$ to the orthogonal complement of $\mathcal{P}$ with respect to the metric $G_3 = g - bg^{-1}b$. Therefore we have $K_+(\mathcal{P}) = K_-(\mathcal{P})$. □

Remark 7.8. As an aside, we mention that for any generalized almost complex structure, $\mathcal{J}$ there is another one $\mathcal{J}'$ such that $(\mathcal{J}, \mathcal{J}')$ are a generalized almost Kähler structure.

Since we will not be using this and since the proof precisely mimics the proof that every almost symplectic manifold has a compatible almost complex structure we do not include the proof here.

The next requirement that one should want to place on $(\mathcal{J}, \mathcal{J}', \mathcal{P})$ is that the distribution $\mathcal{Q}$ be involute. We plan to return to this analysis in a future paper. This is the analogue of considering a flat connection in definition of the term *semi-flat* in Section 2 of Part II [1].

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